1. PHD PROJECT DESCRIPTION (4000 characters max., including the aims and work plan)

Project title: ***Backward stochastic differential equations and their applications to optimization problems***

1.1. Project goals

The goal of the project is to provide a series of existence and uniqueness results for Backward Stochastic Differential Equations (BSDEs), Reflected BSDEs (RBSDEs), second order BSDEs and second order RBSDEs, and then to show the relations of solutions of these problems with value functions in various optimization problems (see Outline) which, in a natural manner, arises in mathematical finance, economy and engineering. Some of these relations are already known and we continue to study on them, whereas some are poorly understood and create challenging problems in the project. As to the optimization problems, we are interested in the existence, regularity, stability and approximation of the value functions, optimal and ε-optimal strategies. We also pay special attention to Markov models and connections of the value functions with Hamilton-Jacobi-Bellman partial differential equations (viscosity and mild solutions).

1.2. Outline

Roughly speaking in an optimization problem we have a set of strategies/choices which determine the outcome (payoff). The goal is to find an optimal strategy/choice which maximizes our payoff. In 1990, Pardoux and Peng introduced in [25] the notion of nonlinear BSDEs on a probability space with filtration generated by a given Wiener process. Independently, in 1992 Duffie and Epstein [8, 9] presented a BSDEs formulation of recursive utility. In 1995-1997, El Karoui et al. [11] and Cvitanic and Karatzas [5] generalized the notion of BSDEs and introduced reflected BSDEs. In both papers, the authors provided formulas that link RBSDEs with optimization problems. Nowadays, the theory of (Reflected) BSDEs and their numerous variants are recognized as very powerful tools in many branches of mathematics, especially in optimization problems and PDEs (see Literature).

The aim of the project is to develop the theory of BSDEs and RBSDEs towards applications in some crucial optimization problems. We will study selected problems from the following topics:

1) BSDEs, second order BSDEs (see [31]) on general filtered spaces and control problems.

2) Reflected BSDEs on general filtered spaces (see [13,15,18,20,29]) and nonlinear stopping problems - the strategy of the player is to stop the game in an optimal moment (stopping time) to maximize the pay-off function, which is described via nonlinear expectation operator (see [12]);

3) BSDEs with two reflecting barriers on general filtered spaces (see [3,10,14,18,19,23]) and nonlinear zero-sum Dynkin games (see [10,14,19]). In this optimization problem we have two players with strategies consisting of stopping times. When player 1 stops the game at time τ before player 2, then player 1 gains reward $A(\tau)$ described via nonlinear expectation of a stochastic process, and player 2 loses this amount (zero-sum game). Otherwise, player 1 gains $B(\tau)$ and player 2 loses $B(\tau)$;

4) Nonzero-sum Dynkin games problem (see [16]). It is the game similar to the one described in 3), but with the considerable change: when player 1 gains $A(\tau)$, then player 2 does not necessarily lose $A(\tau)$, and when player 1 gains $B(\tau)$, then player 2 does not necessarily lose $B(\tau)$. An open problem is the form of the associated BSDE;

5) Second order reflected BSDEs (see [24]) and robust optimal stopping problems and robust Dynkin games (see [1]). In these problems, the strategies of the player/players consist of stopping times and family of probability measures (uncertainty) describing the gain functional;

6) Quasi-linear reflected BSDEs and impulse control problems (see [7]), where the strategy of the player consists of an increasing sequence (τ_n) of stopping times (intervention times) and a sequence (c_n) of appropriately measurable random variables (cost of the intervention). At any time τ_n the player may shift the process with some cost c_n (so called control action);

7) Multidimensional reflected BSDEs (see [2]) in random convex and non-convex closed sets and the switching problem (see [17]), where the strategy of the player is again an increasing sequence (τ_n) of stopping times. In this case at any time τ_n the player switches (with some cost c_n) to the different cost/profit process (generator).

1.3. Work plan

See Sections 1.1, 1.2.

1.4. Literature

1. Bayraktar, E., Yao, S.:On the robust Dynkin game. Ann. Appl. Probab. 27 (2017), no. 3, 1702-1755.
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4. Buckdahn, R., Li, J., Peng, S., Rainer, C.: Mean-field stochastic differential equations and associated PDEs. Ann. Probab. 45 (2017), no. 2, 824-878.
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1.5. **Required initial knowledge and skills of the PhD candidate:** analytic thinking, willingness of self-study, understanding of mathematical analysis, solid knowledge on probability theory, functional analysis, and basic knowledge on stochastic processes, differential equations.

1.6. **Expected development of the PhD candidate's knowledge and skills:** conducting research on a high level, acquiring skills to present scientific achievements on professional level, acquiring advanced knowledge on stochastic equations, stochastic processes, optimization problems and stochastic analysis.