1. **PHD PROJECT DESCRIPTION (4000 characters max., including the aims and work plan)**
 Project title:

Periodic solutions of symmetric Hamiltonian systems

1.1. **Project goals:** Generally speaking, the first aim of the project is to study the existence and properties of critical orbits of invariant fuctionals.
Next, applying these abstract results we are planning to study closed connected sets of nonstationary periodic solutions of symmetric Hamiltonian systems in a neighborhood of orbits of equilibria, continuation of such sets and their global bifurcations.
Finally, we are also going to apply our abstract results to problems coming from natural sciences.

1.2 **Outline:** More precisely, we propose to consider a Hamiltonian system

$$\dot{x}(t) = JH'(x(t)), \tag{0.1}$$ hs

where $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ and J is the standard symplectic matrix.
In our research we use the variational approach, i.e. we introduce a parametrized family of Hamiltonian systems

$$\dot{x}(t) = \lambda JH'(x(t)), \tag{0.2}$$ hsf

and consider 2π-periodic solutions of this family as critical points of an associated functional $\Phi : H^{1/2}(S^1, \mathbb{R}^{2N}) \times (0, +\infty) \to \mathbb{R}$, where $H^{1/2}(S^1, \mathbb{R}^{2N})$ is the Sobolev space of 2π-periodic $\mathbb{R}^{2N}$-valued functions and

$$\Phi(x, \lambda) = \frac{1}{2} \int_0^{2\pi} (J\dot{x}(t) \cdot x(t))dt + \int_0^{2\pi} \lambda H(x(t))dt.$$

It is known that the symmetries of the problem are inherited by the functional. In particular, since the space $H^{1/2}(S^1, \mathbb{R}^{2N})$ consists of 2π-periodic functions, the functional Φ is, by definition, S^1-invariant, where the S^1-action on $H^{1/2}(S^1, \mathbb{R}^{2N})$ is given by shift in time. If we additionally assume that $\mathbb{R}^{2N}$ is a symplectic, orthogonal representation of some compact Lie group Γ and the Hamiltonian H is Γ-invariant, then Φ is $(\Gamma \times S^1)$-invariant.
In other words the study of 2π-periodic solutions of the family (0.2) is equivalent to the study of critical $(\Gamma \times S^1)$-orbits of solutions of the equation $\nabla_x \Phi(x, \lambda) = 0$.
To study the critical points of the functional Φ, taking into account the $(\Gamma \times S^1)$-symmetries of the problem, we are going to apply equivariant counterparts of known tools i.e. the $(\Gamma \times S^1)$-equivariant Conley index, and the degree for $(\Gamma \times S^1)$-equivariant gradient maps.
First, **we intend to study** the existence, bifurcations (local and global) and continuation of nonstationary periodic solutions of autonomous symmetric Hamiltonian systems in a neghborhoods of Γ-orbits od stationary points.. Next, **we are planning to study** nonstationary periodic solutions of specific Hamiltonian systems that arise in celestial mechanics.
To be more precise.
Consider the following family of Hamiltonian systems

$$\begin{cases} \dot{x}(t) &= \lambda JH'(x(t)) \\ x(0) &= x(2\pi) \end{cases} . \tag{0.3}$$ fhams

Let $H'^{-1}(0)$ consists of a finite number of orbits, i.e. there are $x_0, x_1, \ldots, x_p \in \mathbb{R}^{2N}$ such that $H'^{-1}(0) = \Gamma(x_0) \cup \Gamma(x_1) \cup \ldots \cup \Gamma(x_p)$. Note that these orbits can be manifolds of different dimensions.
The set of stationary solutions of the family (0.3) equals

$$\mathcal{S} = \left(\bigcup_{i=0}^{p} \Gamma(x_i) \right) \times (0, +\infty) \subset H^{1/2}(S^1, \mathbb{R}^{2N}) \times (0, +\infty)$$

We intend to explore global bifurcations of nonstationary 2π-periodic solutions of the family (0.3) from the set $\mathcal{S}$ i.e. we are going to study properties of closed connected sets of nonstationary 2π-periodic solutions of this family emanating from the set $\mathcal{S}$.
According to the knowledge of the author of the project, this type of problems **has been studied so far only** if the set of critical points of the Hamiltonian H has been finite.
The importance of abstract results obtained during the implementation of the research project can be emphasized by using them to the study of nonstationary periodic solutions of specific Γ-symmetic Hamiltonian systems.
Applying the abstract results to symmetric Hamiltonian systems we will consider the following $SO(2)$-symmetric Hamiltonian systems coming from celestial mechanics:

(i) Meyer and Schmidt gave a simple mathematical model for braided rings of a planet based on Maxwell's model for the rings of Saturn. The authors consider Nonalternating $(N+1)$-body problem and Alternating $(2N+1)$-body problem. After some reductions in both cases the Hamiltonians are $\Gamma = SO(2)$-invariant. Moreover, the sets of stationary solutions these $SO(2)$-symmetric Hamiltonian systems consist of a one-dimensional manifold homeomorphic to S^1.

(ii) Palacian, Vidal, Vidarte and Yanguas considered a family of cosmology-inspired Hamiltonian systems depending on three paramaters $\alpha, \beta \in \mathbb{R}$ and $\epsilon > 0$. If $\alpha = \beta < 0$ then this Hamiltonian is $\Gamma = SO(2)$-symmetric. Moreover, the sets of stationary solutions these $SO(2)$-symmetric Hamiltonian system consists of the origin and a one-dimensional manifold homeomorphic to S^1.

The above presented Hamiltonian systems are $SO(2)$-invariant **We expect to prove** the existence, bifurcation and continuation of new nonstationary periodic solutions to these issues in a neighbourhood of the $SO(2)$-orbits of equilibria.

1.3 **Work plan:** The research project consists of two closely related topics. **The first one** is the study of nonstationary periodic solutions of symmetric Hamiltonian systems. **The second one** is the study of periodic solutions of specific Hamiltonian systems appearing in celestial mechanics.
The general study plan is as follows:

(i) studying properties of the Euler ring $U(\Gamma \times S^1)$,

(ii) applications of $(\Gamma \times S^1)$-equivariant Conley index and the degree theory for $(\Gamma \times S^1)$-equivariant gradient maps to the study of $(\Gamma \times S^1)$-orbits of critical point of $(\Gamma \times S^1)$-invariant functionals,

(iii) studying Γ-symmetric Hamiltonian systems:

  (A) local and global bifurcations of nonstationary periodic solutions,

  (B) continuation of nonstationary periodic solutions.

(iv) Applications of the abstract results to the study of nonstationary periodic solutions of Hamiltonian systems modelling
   (A) Saturn rings motion,
   (B) homogeneous and isotropic expanding/contracting of universe.
(v) Comparison of the obtained results with those existing in the literature.

1.4 **Literature** (max. 10 listed, as a suggestion for a PhD candidate)

(1) A. Gołębiewska, E. Pérez-Chavela, S. Rybicki, A. Ureña, Bifurcation of closed orbits from equilibria of Newtonian systems with Coriolis forces, Journal of Differential Equations 338 (2022), 441-473
(2) K.R. Meyer. G.R. Hall & D. Offin, *Introduction to Hamiltonian dynamical systems nd the N-body problem,* Applied Mathematical Sciences **90**, Springer, 2009,
(3) M. Izydorek, *Equivariant Conley index in Hilbert spaces and applications to strongly indefinite problems*, Nonlinear Analysis TMA **51(1)** (2002), 33-66,
(4) E. N. Dancer & S. Rybicki, *A note on periodic solutions of autonomous Hamiltonian systems emanating from degenerate stationary solutions*, Differential and Integral Equations **12(2)** (1999), 147-160,
(5) M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, 1996,
(6) K. Gęba, *Degree for gradient equivariant maps and equivariant Conley index*, Topological Nonlinear Analysis, Degree, Singularity and Variations, (Ed.) M. Matzeu i A. Vignoli, PNDLE **27**, Birkhäuser, (1997), 247-272,
(7) A. Gołębiewska & S. Rybicki, *Global bifurcations of critical orbits of G-invariant strongly indefinite functionals*, Nonlinear Analysis TMA **74(5)** (2011), 1823-1834,
(8) Shui-Nee Chow, Ch. Li & D. Wang, *Normal Forms and Bifurcations of Planar Vector Fields,* Cambridge University Press 1994,
(9) J. Mawhin & M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, Berlin Heidelberg New York, 1989,
(10) P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Regional Conference Series in Mathematics **65**, AMS, 1985.

1.5 **Required initial knowledge and skills of the PhD candidate:** The candidate should have the ability to think independently, analytically, willingness to self-study and mathematical inquisitiveness. They should also have basic knowledge and skills in the field of mathematical analysis, functional analysis, linear algebra, topology, topological nonlinear analysis and ordinary differential equations.

1.6 **Expected development of the PhD candidate's knowledge and skills:** While implementing the research project, the candidate should conduct research at a high level, acquire the skills to present scientific achievements at a professional level, gain advanced knowledge of critical points theory, invariant topological nonlinear analysis, and theory of Hamiltonian and Newtonian systems.